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WIENER–WINTNER RETURN-TIMES ERGODIC THEOREM

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ABSTRACT

We state a new ergodic theorem, combining the Wiener–Wintner theorem and Bourgain's theorem concerning the convergence of ergodic averages along return-times sequences.

We consider ergodic averages of the form

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta}\cdot f'(S^n y)\cdot f(T^n x)$$

and we show that the behaviour of these averages characterizes an algebra $\mathcal{C}$ of functions, which contains the Kronecker algebra and has interesting properties, linked with multiple recurrence ergodic theorems.

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Introduction

Let $(X, \mathcal{F}, \mu, T)$ be a probability measure preserving system (**m.p.s.**). The Wiener–Wintner ergodic theorem, respectively the Bourgain return-times theorem, states that, for all $f \in L^\infty(\mu)$ and for μ-almost all x, the sequence $(f(T^n x))_{n\geq 0}$ is a good sequence of weights for the weighted mean ergodic theorem, respectively for the weighted pointwise ergodic theorem.

Let us recall precisely these two results.

WIENER–WINTNER THEOREM: *If $(X, \mathcal{F}, \mu, T)$ is a m.p.s. and if $f \in L^1(\mu)$, then for μ-almost all x, for all $\theta \in \mathbb{R}$, the sequence*

$$\left(\frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta} f(T^n x)\right) \tag{1}$$

converges.

Moreover, if the m.p.s. is ergodic, there is an equivalence between

- *the function f is orthogonal to the Kronecker factor of the m.p.s. (i.e., orthogonal to all the eigenfunctions for the action of T on $L^2(\mu)$);*
- *for μ-almost all x,*

$$\lim_{N\to+\infty} \sup_{\theta\in\mathbb{R}} \left|\frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta}\cdot f(T^n x)\right| = 0\,;$$

- *for all $\theta \in \mathbb{R}$,*

$$\lim_{N\to+\infty} \left\|\frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta}\cdot f\circ T^n\right\|_1 = 0\,.$$

This result first appears in [WW] for its first part. The uniform result is used in [B2] and its proof can be found in, for example, [A1] or [L2].

BOURGAIN'S RETURN-TIMES THEOREM: *If $(X, \mathcal{F}, \mu, T)$ is a m.p.s. and if $f \in L^2(\mu)$, then there is a set of full measure $X(f) \subset X$ such that , for all $x \in X(f)$, for any m.p.s. (Y, φ, ν, S), and any $f' \in L^2(\nu)$, the sequence*

$$\left(\frac{1}{N}\sum_{n=0}^{N-1} f'(S^n y)\cdot f(T^n x)\right) \tag{2}$$

converges for ν-almost all y.

Proofs of this result can be found in [B1], [BFKO] and in [R1]. It is not difficult to see that the first part of the Wiener–Wintner theorem is exactly the version "convergence in $L^2(\nu)$" of this ergodic theorem.

We now state our convergence theorem.

THEOREM 1: *Let $(X,\mathcal{F},\mu,T)$ be a m.p.s. and $f \in L^2(\mu)$. For μ-almost all x we have: for any m.p.s. (Y,φ,ν,S) and any $f' \in L^2(\nu)$, for ν-almost all y, for all $\theta \in \mathbb{R}$, the sequence*

$$\left(\frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta}\cdot f'(S^n y)\cdot f(T^n x)\right) \tag{3}$$

converges.

Remarks:

- The fact that, if $(X,\mathcal{F},\mu,T)$ and (Y,φ,ν,S) are m.p.s., $f \in L^1(\mu)$ and $f' \in L^1(\nu)$, then, for all $\theta \in \mathbb{R}$, for ν-almost all y, for μ-almost x, the sequence (3) converges is a simple consequence of Birkhoff's ergodic theorem.
- The necessity of taking f and f' square integrable in the return-times theorem is discussed in [A2].
- If $(X,\mathcal{F},\mu,T)$ is weakly mixing Theorem 1 is a simple consequence of Bourgain's return-times theorem as shown in [A2].

In the same manner that the averages (1) are related to the Kronecker factor of the system, we shall see that the averages (3) are related to a new factor of the system. This factor first appears in the work of Conze and the second author concerning the convergence of ergodic averages of the form $\frac{1}{N}\sum_{n=0}^{N-1} f(T^{pn}x)\cdot g(T^{qn}x)\cdot h(T^{rn}x)$, where p, q and r are integers (cf. [CL1], [CL2]). This factor also appears in the study of convergence of more general Furstenberg's multiple recurrence ergodic averages for distal systems (cf. [L1]).

In [R2] a detailed study of this factor is achieved, and it is described how this factor is related to Kronecker factors of the ergodic components of product measures. This study will be used in the sequel.

The algebra of functions, in $L^\infty(\mu)$, measurable with respect to this factor of $(X,\mathcal{F},\mu,T)$ will be called $\mathcal{C}$ $(= \mathcal{C}(X,\mathcal{F},\mu,T))$ and its definition is given in Section 1. This algebra is characterized by the next theorem.

THEOREM 2: *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic m.p.s. and $f \in L^2(\mu)$. There is an equivalence between:*

(i) *the function f is orthogonal to $\mathcal{C}$;*

(ii) *for μ-almost all x, for any m.p.s. (Y, φ, ν, S) and any $f' \in L^2(\nu)$, for ν-almost all y,*

$$\lim_{N\to+\infty} \sup_{\theta\in\mathbb{R}} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} \cdot f'(S^n y) \cdot f(T^n x) \right| = 0\,;$$

(iii) *for $\mu \otimes \mu$-almost all (x, x'), for all $\theta \in \mathbb{R}$,*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} \cdot \bar{f}(T^n x') \cdot f(T^n x) = 0\,;$$

(iv)

$$\lim_{N\to+\infty} \left\| \sup_{\theta\in\mathbb{R}} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} \cdot \bar{f}(T^n x') \cdot f(T^n x) \right| \right\|_{L^2(\mu\otimes\mu)} = 0\,.$$

In the proof of these theorems we need to describe some disintegrations of measures associated with the return-times theorem. In order to do this, it is convenient to work with regular m.p.s. A m.p.s. $(X, \mathcal{F}, \mu, T)$ is called **regular** if the space X is compact metrizable, the σ-algebra $\mathcal{F}$ is Borel and the transformation T is continuous. It is known that every separable m.p.s. is equivalent to a regular one (cf., for example, [F], chapter 5), and it is enough for us to prove our theorems for regular m.p.s.

Our paper is organized as follows. In the first section we give the definition of the factor $\mathcal{C}$: it is the maximal factor of the type "abelian group extension of a group rotation" which satisfies a certain functional equation. Then we give another form of a result in [R2] on eigenfunctions of product transformations.

In Section 2, we prove a disintegration measure theorem related to the return-times theorem.

The third section is devoted to the proof of Theorem 1 for functions f of the algebra $\mathcal{C}$. We used here the functional equation and the Wiener–Wintner theorem.

In Section 4, we prove theorem 1 for functions f in the orthogonal of $\mathcal{C}$. We used here Bourgain's return-times theorem and the classical Van der Corput inequality.

Theorem 2 is proved in the last section.

1. Eigenfunctions of product transformations and the algebra $\mathcal{C}$

We recall, for the reader's convenience, the definition of the algebra $\mathcal{C}$ and some basic facts about it. Details of the matters given in this section can be found in [R2].

1.1 DEFINITION. Let $(X, \mathcal{F}, \mu, T)$ be an ergodic m.p.s.

This system admits a maximal factor of the type "extension by a compact abelian group of a translation on a compact abelian group", that is to say a system $(G \times G_1,\ B(G \times G_1),\ m_G \otimes m_{G_1}, R_{g_0,\varphi})$, where

G and G_1 are compact abelian groups,
$B(\cdot)$ is the borelian σ-algebra and $m_.$ the Haar measure,
φ is a measurable map from G into G_1,
$g_0 \in G$,
$R_{g_0,\varphi}, (g, g_1) = (g_0 g, \varphi(g) g_1)$, and
$(G, B(G), m_G, R_{g_0}\colon g \mapsto g g_0)$ is the maximal discrete spectrum factor.

A character $\mathcal{X}$ of G_1 will be called a $\mathcal{C}$-**character** if there exists measurable maps $\lambda\colon G \to S^1$ and $u\colon G \times G \to S^1$ such that, for all $t \in G$, for a.a. $g \in G$,

$$\text{(E)} \qquad \frac{\mathcal{X}(\varphi(tg))}{\mathcal{X}(\varphi(g))} = \lambda(t) \frac{u(t, g_0 g)}{u(t, g)}.$$

The set of $\mathcal{C}$-characters is a subgroup of the dual group $\hat{G}_1$; we denote by K its orthogonal. The system $(G \times G_1, \ldots, R_{g_0,\varphi})$ has a natural factor $(G \times G_1/K, B(G \times G_1/K), m_G \otimes m_{G_1/K}, R_{g_0,\varphi})$. This factor will be called the $\mathcal{C}$-**factor**, and we denote by $\mathcal{C}$ the algebra of $\mathcal{C}$-factor measurable bounded functions on X.

1.2 EXAMPLES. The two basic examples of m.p.s. which coincide with their $\mathcal{C}$-factor are given by:

- Extensions by compact abelian groups of translations on compact abelian groups with affine cocycles φ.
- Translations on compact quotients of order two nilpotent groups.

1.3 DISINTEGRATION OF PRODUCT MEASURES. Let $(X, \mathcal{F}, \mu, T)$ and (Y, φ, ν, S) be two ergodic m.p.s.

These two systems admit a maximal common discrete spectrum factor, which can be represented as a translation on a compact abelian group

$$(H, B(H), m_H, R_{h_0}) \quad \text{with } R_{h_0}(h) = h_0 h\,.$$

We denote by E_H^X and E_H^Y the projection operators (conditional expectations) defined on, respectively, $\mathcal{F}$ and $\mathcal{G}$, with values in $L^\infty(H)$. For a.a. c in H, we can consider the measure $(\mu \otimes \nu)_c$ on $X \times Y$ defined by

$$\text{if } A \in \mathcal{F} \text{ and } B \in \mathcal{G}, \quad (\mu \otimes \nu)_c(A \times B) = \int_H (E_H^X(A))(h) \cdot (E_H^Y(B))(ch)dh\,.$$

In other words, the measure $(\mu \otimes \nu)_c$ is the relatively independent joining of μ and ν above the common factor $(\{(h, ch) |\ h \in H\},\ R_{h_0} \times R_{h_0})$. It is known that these measures $(\mu \otimes \nu)_c$ are $T \times S$-ergodic and that $\mu \otimes \nu = \int_H (\mu \otimes \nu)_c dc$.

1.4 KRONECKER FACTORS OF ERGODIC COMPONENTS. We denote by $(G \times G_2,\ B(G \times G_2),\ m_G \otimes m_{G_2}, R_{g_0,\varphi})$ the $\mathcal{C}$-factor of the ergodic m.p.s. $(X, \mathcal{F}, \mu, T)$, and by $(G' \times G_2',\ B(G' \times G_2'),\ m_{G'} \otimes m_{G_2'}, R_{g_0',\varphi'})$ the $\mathcal{C}$-factor of the ergodic m.p.s. (Y, φ, ν, S). There are closed subgroups L and L' of, respectively, G and G', such that $H = G/L = G'/L'$, and $h_0 \in H$ such that $h_0 = g_0 L = g_0' L'$. Let s be a measurable section from H into G (for each $h \in H$, $s(h)L = h$).

The m.p.s. $(G, \mathcal{B}(G), m_G, R_{g_0})$ can be described under the form

$$(H \times L, \mathcal{B}(H) \times \mathcal{B}(L), m_H \times m_L, R_{h_0,\sigma}) \qquad \text{where } \sigma(h) = s(h_0 h)^{-1} g_0 s(h).$$

In the same way, let s' be a measurable section from H into G', and σ' the associated cocycle from H into G'.

For each $c \in H$, we consider the m.p.s.

$$\mathcal{C}_c = (H \times L \times L' \times G_2 \times G_2',\ \text{Borel } \sigma\text{-algebra, Haar probability, } U_c)$$

where

$$U_c(h, l, l', g_2, g_2',) = (h_0 h, \sigma(h) l, \sigma'(ch) l', \varphi(s(h) l) g_2, \varphi'(s'(ch) l') g_2')).$$

For almost all c, the m.p.s. $\mathcal{C}_c$ is a factor of the m.p.s. $(X \times Y, (\mu \otimes \nu)_c, T \times S)$.

In [R2] the next theorem is proved.

THEOREM 3: *For almost all c in H, the eigenfunctions of the m.p.s. $(X \times Y, (\mu \otimes \nu)_c, T \times S)$ are $\mathcal{C}_c$-measurable.*

In the sequel we will use this result in the following form:

THEOREM 3′: *Let $f \in L^2(\mu)$ and $f' \in L^2(\nu)$. If the function f is orthogonal to the C-factor of the m.p.s. $(X, \mathcal{F}, \mu, T)$ then, for almost all c in H, the function $f \otimes f'$ is orthogonal to the Kronecker factor of the m.p.s. $(X \times Y, (\mu \otimes \nu)_c, T \times S)$.*

A converse of Theorem 3′ is given by Proposition 3 in Section 5.

2. Fubini Return-Times Theorem

The disintegration of product measure presented in the next theorem reduces to the Fubini theorem in the case where $S = I$.

If X is a topological space, we denote by $C(X)$ the space of continuous complex functions on X.

THEOREM 4: *Let $(X, \mathcal{F}, \mu, T)$ be a regular ergodic m.p.s.. There exists a set of full measure $\widetilde{X} \subset X$ such that*

(1) *for all $x \in \widetilde{X}$, for any regular m.p.s. (Y, φ, ν, S), there exists a family $(m_{xy})_{y \in Y}$ of $T \times S$-invariant probability measures on $X \times Y$ such that*

(1-a) *for ν-almost all y, for all $f \in C(X)$ and all $f' \in C(Y)$,*

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f'(S^n y) f(T^n x) = \int_{X \times Y} f \otimes f' \, dm_{xy} \,;$$

(1-b)

$$\mu \otimes \nu = \int_Y m_{xy} \, d\nu(y);$$

(2) *if $f \in L^1(\mu)$, there exists a set of full measure $\widetilde{X}_f \subset \widetilde{X}$ such that, for all $x \in \widetilde{X}_f$, we have:*
for any regular m.p.s. (Y, φ, ν, S), for all $f' \in C(Y)$, for ν-almost all y,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f'(S^n y) f(T^n x) = \int_{X \times Y} f \otimes f' \, dm_{xy} \,.$$

Remark: The hypothesis of ergodicity of the m.p.s. $(X, \mathcal{F}, \mu, T)$ is crucial in this theorem.

Proof of Theorem 4: Let D be a denumerable set of continuous functions on X, dense in $C(X)$ for the uniform topology. With each f in D we can associate, by the return-times theorem, a universal set of full measure $X(f)$.

We denote by X' the set of generic points of the m.p.s. $(X, \mathcal{F}, \mu, T)$; for all $f \in C(X)$, and all $x \in X'$

$$\lim \frac{1}{N} \sum_{n<N} f(T^n x) = \int_X f d\mu .$$

We denote by $\widetilde{X}$ the set $X' \cap (\bigcap_{f \in D} X(f))$. This set $\widetilde{X}$ is of full measure in X. We remark that the set $\widetilde{X}$ is, for all continuous functions f on $C(X)$, a universal set in the sense of the return-times theorem. Let us quickly see why. We have, for all $f \in C(X)$, all $\tilde{f} \in D$, and all $x \in \widetilde{X}$,

$$\begin{aligned} &\left| \overline{\lim} \frac{1}{N} \sum_{n<N} f'(S^n y) f'(T^n x) - \underline{\lim} \frac{1}{N} \sum_{n<N} f'(S^n y) f(T^n x) \right| \\ &\quad = \left| \overline{\lim} \frac{1}{N} \sum_{n<N} f'(S^n y)(f - \tilde{f})(T^n x) - \underline{\lim} \frac{1}{N} \sum_{n<N} f'(S^n y)(f - \tilde{f})(T^n x) \right| \\ &\quad \leq 2 \| f - \tilde{f} \|_\infty \lim \frac{1}{N} \sum_{n<N} |f'|(S^n y) \end{aligned}$$

(for any m.p.s. (Y, φ, ν, S) and all $f' \in L^1(\nu)$).

We now fix $x \in \widetilde{X}$. Let (Y, φ, ν, S) be a regular m.p.s., and D' a denumerable dense set in $C(Y)$.

With each f in D and each f' in D', we associate a set $Y(x, f, f')$ of full measure in Y, where the sequence $\left(\frac{1}{N} \sum_{n<N} f'(S^n y) f(T^n x)\right)$ converges. We use the notation $Y(x) = \bigcap \{Y(x, f, f') |\ f \in D \text{ and } f' \in D'\}$. This set $Y(x)$ is of full measure in Y and, by the same argument as before, we have: for all $f \in C(X)$ and $f' \in C(Y)$, the sequence $(\frac{1}{N} \sum_{n<N} f'(S^n y) f(T^n x))$ converges for all $y \in Y(x)$.

Let B be the algebra of finite linear combination of functions on $X \times Y$ of the form $f \otimes f'$ with $f \in C(X)$ and $f' \in C(Y)$. For each $h \in B$, we note

$$L_{xy}(h) = \lim \frac{1}{N} \sum_{n<N} h(T^n x, S^n y) .$$

This defines (for each $y \in Y(x)$) a positive uniformly continuous linear form on B. It extends by continuity to all $C(X \times Y)$. We have $L_{xy}(1) = 1$, thus this extension defines a probability measure m_{xy} on $X \times Y$ satisfying 1-a. Because of the continuity of T and S, it is clear that this measure is $T \times S$-invariant.

Let us now prove 1-b. We keep x fixed in $\widetilde{X}$. If $f \in C(X)$ and $f' \in C(Y)$, the map $y \mapsto L_{xy}(f \otimes f')$ is bounded measurable, as limit of a sequence of uniformly bounded continuous functions, and

$$\begin{aligned}\int_Y L_{xy}(f \otimes f')d\nu(y) &= \int \lim_N \Big(\frac{1}{N} \sum_{n<N} f'(S^n y) f(T^n x)\Big) d\nu(y) \\ &= \lim_N \frac{1}{N} \sum_{n<N} f(T^n x) \int f' d\nu \\ &= \int f \otimes f' d\mu \otimes \nu\end{aligned}$$

because x is generic.

This equality now easily extends to all functions $h \in C(X \times Y)$ and so we have reached the identity

$$\int_{X \times Y} \int_Y h(x', y') dm_{xy}(x', y') d\nu(y) = \int_{X \times Y} h(x', y') d\mu(x') d\nu(y') \tag{4}$$

for any $x \in \widetilde{X}$ and $y \in Y(x)$.

It remains to prove that, for all $F \in L^1(\mu \otimes \nu)$, we have: for ν-almost all y, $F \in L^1(m_{xy})$; the map $y \mapsto \int_{X \times Y} F dm_{xy}$ is ν-integrable and

$$\iint F dm_{xy} d\nu(y) = \int F d(\mu \otimes \nu).$$

Going from (4) to this point is a general result on measure disintegrations and it can be done by classical arguments used in the proof of the Fubini theorem. Let us sketch these arguments in our particular case.

We fix x in $\widetilde{X}$. In a first step, we prove that if N is a $\mu \otimes \nu$-negligible set then, for ν-almost y, $m_{xy}(N) = 0$. In order to do this it suffices to write the characteristic function 1_N as a decreasing limit of lower semi-continuous functions and each of these lower semi-continuous functions as an increasing limit of continuous ones, and to apply (4). In the second step we consider a function

F in $L^1(\mu \otimes \nu)$. There exists a sequence (u_n) of continuous functions on $X \times Y$ such that

$$\sum_n u_n = F \quad \text{in } L^1(\mu \otimes \nu) \quad \text{and} \quad \sum_n \|u_n\|_1 < +\infty .$$

We have $\sum_n u_n = F$ outside a $\mu \otimes \nu$-negligible set N. By (4), we have

$$\sum_n \int \left(\int |u_n| dm_{xy} \right) d\nu(y) = \sum_n \int |u_n| d\mu \otimes \nu < +\infty . \tag{5}$$

So, for ν-a.a. y, $\sum_n \int |u_n| dm_{xy} < +\infty$. This implies that, for ν-a.a. y, $\sum_n u_n$ converges outside a m_{xy}-negligible set N_y. So, for ν-a.a. y, $F = \sum_n u_n$ outside the m_{xy}-negligible set $N_y \cup N$. (Here we use the first step.)

We also have

$$\int F dm_{xy} = \sum_n \int u_n dm_{xy} .$$

Coming back to (5), we have $\sum_n \int |\int u_n dm_{xy}| d\nu(y) < +\infty$, which implies that $\sum_n \int u_n dm_{xy}$ converges in $L^1(\nu)$ and

$$\int \left(\sum_n \int u_n dm_{xy} \right) d\nu(y) = \sum_n \int \left(\int u_n dm_{xy} \right) d\nu .$$

Using (4), this is exactly

$$\iint F dm_{xy} d\nu(y) = \sum_n \int u_n d\mu \otimes \nu = \int F d\mu \otimes \nu .$$

We have proved the first part of Theorem 4. We now prove the second.

Let $f \in L^1(\mu)$. We fix a sequence (f_k) of continuous functions on X which converges to f in $L^1(\mu)$. We denote by $\widetilde{X}_f$ the set of all x in X such that

$x \in \widetilde{X}$ and

for each k, $\lim_N \frac{1}{N} \sum_{n<N} |f - f_k|(T^n x) = \|f - f_k\|_1$.

This set $\widetilde{X}_f$ is of full measure. We fix $x \in \widetilde{X}_f$. If (Y, φ, ν, S) is a regular m.p.s. and $f' \in C(Y)$, we have, for ν-a.a. y,

$$\begin{aligned}
\limsup_N & \left| \frac{1}{N} \sum_{n<N} f'(S^n y) f(T^n x) - \int_{X \times Y} f \otimes f' dm_{xy} \right| \\
&\leq \left(\limsup_N \frac{1}{N} \sum_{n<N} |f'|(S^n y) |f - f_k|(T^n x) \right) + \int |f - f_k| \otimes |f'| dm_{xy} \\
&\leq \|f'\|_\infty \|f - f_k\|_1 + \int |f - f_k| \otimes |f'| dm_{xy}
\end{aligned}$$

(the first of these two inequalities makes use of 1-a, and the second uses the choice of x).

By 1-b we then have

$$\int \limsup_N \left| \frac{1}{N} \sum_{n<N} f'(S^n y) f(T^n x) - \int_{X\times Y} f \otimes f' dm_{xy} \right| d\nu(y)$$
$$\leq \|f'\|_\infty \|f - f_k\|_1 + \|f - f_k\|_1 \|f'\|_1 .$$

This implies that, ν-a.a. y,

$$\lim_N \frac{1}{N} \sum_{n<N} f'(S^n y) f(T^n x) = \int_{X\times Y} f \otimes f' dm_{xy} ,$$

and Theorem 4 is proved.

3. Proof of Theorem 1 for functions f in the algebra $\mathcal{C}$

PROPOSITION 1: *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic m.p.s. The set of functions f in $L^2(\mu)$ such that*

for μ-almost all x, for any m.p.s. (Y, φ, ν, S), for all $f' \in L^2(\nu)$,
for ν-almost all y, and for all $\theta \in \mathbb{R}$
the sequence $\frac{1}{N}\sum_{n<N} e^{in\theta} f'(S^n y) f(T^n x)$ converges

is a closed linear subspace of $L^2(\mu)$.

Proof of Proposition 1: It is easy to see that if (f_m) is a sequence in $L^2(\mu)$ which converges to f, and if an element x of X satisfies, for each m, the sequence

$$\left(\frac{1}{N} \sum_{n<N} e^{in\theta} f'(S^n y) f_m(T^n x) \right),$$

converges for ν-a.a. y and

$$\lim_N \frac{1}{N} \sum_{n<N} |f - f_m|^2 (T^n x) = \|f - f_m\|_2^2 ,$$

then for ν-a.a. y, the sequence $\left(\frac{1}{N}\sum_{n<N} e^{in\theta} f'(S^n y) f_m(T^n x)\right)$ converges for all θ.

The proposition is a direct consequence of this fact.

Remark: In order to prove Theorem 1, it is enough to consider functions f' in a dense subset of $L^2(\nu)$; this is easy to see by looking at bounded functions f (allowed by Proposition 1) and using the maximal ergodic theorem, uniformly in θ.

We now want to prove theorem 1 for functions f in $\mathcal{C}$. Such functions can be seen as defined on the $\mathcal{C}$-factor

$$(G \times G_2,\ B(G \times G_2),\ m_G \otimes m_{G_2}, R_{g_0,\varphi})$$

and, according to Proposition 1, it is enough to prove the result for functions f of the form

$$f(g, g_2) = \gamma(g) \cdot \mathcal{X}(g_2) \quad \text{with } \gamma \in \hat{G} \quad \text{and} \quad \mathcal{X} \in \hat{G}_2 .$$

We fix such a function f. We then have

$$\frac{1}{N} \sum_{n<N} e^{in\theta} \cdot f\left(R^n_{g_0,\varphi}(g, g_2)\right) \cdot f'(S^n y)$$
$$= \left[\frac{1}{N} \sum_{n<N} e^{in\theta} \cdot \gamma(g_0)^n \cdot \mathcal{X}_0\varphi^{(n)}(g) \cdot f'(S^n y)\right] \cdot \gamma(g) \cdot \mathcal{X}(g_2),$$

where $\varphi^{(n)}(g) = \varphi(g) \cdot \varphi(g \cdot g_0) \cdots \varphi(g_0^{n-1} \cdot g)$. So we are looking at the sequence $\frac{1}{N} \sum_{n<N} e^{in\theta} \cdot \gamma(g_0)^n \cdot \mathcal{X}_0\varphi^{(n)}(g) \cdot f'(S^n y)$. We know that there exist measurable maps λ from G into S^1 and u from $G \times G$ into S^1 such that, for any $t \in G$,

$$\frac{\mathcal{X}_0\varphi(gt)}{\mathcal{X}_0\varphi(g)} = \lambda(t) \cdot \frac{u(t, g_0 g)}{u(t, g)} .$$

To exploit this equation, we are going to replace g by gt in the sequence we want to study. We have

$$\frac{\mathcal{X}_0\varphi^{(n)}(gt)}{\mathcal{X}_0\varphi^{(n)}(g)} = \lambda(t)^n \cdot \frac{u(t, g_0^n g)}{u(t, g)}$$

and the sequence $\frac{1}{N} \sum_{n<N} e^{in\theta} \cdot \gamma(g_0)^n \cdot \mathcal{X}_0\varphi^{(n)}(gt) \cdot f'(S^n y)$ converges iff the sequence $\frac{1}{N} \sum_{n<N} e^{in\theta} \cdot \gamma(g_0)^n \cdot \lambda(t)^n \cdot u(t, g_0^n g) \cdot \mathcal{X}_0\varphi^{(n)}(g) \cdot f'(S^n y)$ converges. By the Wiener–Wintner theorem, we have:

$$(5)\quad \begin{cases} \text{for any m.p.s. } (Y, \varphi, \nu, S), \text{ for all } f' \in L^\infty(\nu), \\ \text{for } m_G \times \nu\text{-a.a. } (g, y), \\ \text{for all } \alpha \in \mathbb{R}, \text{ the sequence } \frac{1}{N} \sum_{n<N} e^{in\alpha} \cdot \mathcal{X}_0\varphi^{(n)}(g) \cdot f'(S^n y) \text{ converges} \end{cases}$$

(apply the Wiener–Wintner theorem to the product of the system (Y, φ, ν, S) by the $\mathcal{C}$-factor, and to the function $(y, g, g_2) \mapsto \mathcal{X}(g_2) f'(y)$).

Let $(v_k)_{k \in \mathbb{N}}$ be a sequence of linear combinations of characters of $G \times G$ such that, for almost all $t \in G$,

$$\lim_{k \to +\infty} \|u(t, \cdot) - v_k(t, \cdot)\|_{L^1(G)} = 0 .$$

In order to construct such a sequence (v_k), take a sequence (ω_k) of linear combination of characters $G \times G$, which converges to the function u in $L^1(G \times G)$; by Fubini, the sequence of functions $t \mapsto \|u(t, \cdot) - \omega_k(t, \cdot)\|_{L^1(G)}$ converges to zero in $L^1(G)$. Extract from this sequence an almost everywhere convergent subsequence.

By (5), we have, for any m.p.s. (Y, φ, ν, S), for all $f' \in L^\infty(\nu)$, for a.a. $g \in G$,

(6) $\begin{cases} \text{for } \nu\text{-a.a. } y, \text{ for all } \theta \in \mathbb{R}, \text{ for all } t \in G \text{ and for all } k \in \mathbb{N}, \\ \text{the sequence} \\ \frac{1}{N} \sum_{n<N} e^{in\theta} \cdot (\gamma(g_0))^n \cdot (\lambda(t))^n \cdot v_k(t, g_0^n g) \cdot \mathcal{X}_0 \varphi^{(n)}(g) \cdot f'(S^n y) \\ \text{converges.} \end{cases}$

Let $G(Y, f')$ be the set of g satisfying (6). Let A be the set of couples (t, g) in $G \times G$ such that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n<N} \left| u(t, g_0^n g) - v_k(t, g_0^n g) \right| = \left\| u(t, \cdot) - v_k(t, \cdot) \right\|_{L^1(G)}$$

and

$$\lim_{k \to +\infty} \|u(t, \cdot) - v_k(t, \cdot)\|_{L^1(G)} = 0 .$$

By Birkhoff's ergodic theorem and by the choice of the v_k, this set A has full measure in $G \times G$.

We claim that

(6') $\begin{cases} \text{for any m.p.s. } (Y, \varphi, \nu, S), \text{ for all } f' \in L^\infty(\nu), \text{ for all } g \in G(Y, f'), \\ \text{for } \nu\text{-a.a. } y, \text{ for all } \theta \in \mathbb{R}, \text{ and for all } t \in G \text{ such that } (t, g) \in A, \\ \text{the sequence} \\ \mathcal{A}_N := \frac{1}{N} \sum_{n<N} e^{in\theta} \cdot \gamma(g_0)^n \cdot \lambda(t)^n \cdot u(t, g_0^n g) \cdot \mathcal{X}_0 \varphi^{(n)}(g) \cdot f'(S^n y) \\ \text{converges.} \end{cases}$

Indeed, using (6), we have for any k,

$$\limsup_{N,M\to+\infty} |\mathcal{A}_N - \mathcal{A}_M| \le 2\,\|f'\|_\infty \limsup_{N\to+\infty}\frac{1}{N}\sum_{n<N} |u(t,g_0^n g) - v_k(t,g_0^n g)|\,.$$

Thus, if $(t,g)\in A$, we have

$$\limsup_{N,M\to+\infty} |\mathcal{A}_N - \mathcal{A}_M| \le 2\,\|f'\|_\infty \inf_k \|u(t,\cdot) - v_k(t,\cdot)\|_{L^1(G)} = 0\,.$$

This proves $(6)'$.

Let $B = \{h \in G|$ for any (Y,φ,ν,S), for all $f'\in L^\infty(\nu)$, there exists $g\in G(Y,f')$ and $t\in G$ such that $(t,g)\in A$ and $h = tg\}$.

From the fact that the set $\{(tg,g)|\,(t,g)\in A\}$ is of full measure in $G\times G$, we deduce that the set B is of full measure in G. Finally, if $h\in B$, then, for any (Y,φ,ν,S), for all $f'\in L^\infty(\nu)$, for ν-a.a. y, for all $\theta\in\mathbb{R}$, the sequence

$$\frac{1}{N}\sum_{n<N} e^{in\theta}\cdot\gamma(g_0)^n\cdot(\mathcal{X}_0\varphi)^{(n)}(h)\cdot f'(S^n(y))$$

converges. This is a direct consequence of the preceding calculations and of statement $(6)'$.

This concludes the first part of the proof of Theorem 1. ∎

4. Proof of Theorem 1 for the functions f in the orthogonal of $\mathcal{C}$

In this section we shall consider regular measure preserving systems. Let $(X,\mathcal{F},\mu,T)$ be an ergodic m.p.s. and $f\in L^\infty(\mu)$.

PROPOSITION 2: *For μ-a.a. x, we have:*

if (Y,φ,ν,S) is an ergodic m.p.s. and f' a continuous function on Y,

(a) *for ν-a.a. y, for all $k\in\mathbb{Z}$,*

$$L_k(x,y) := \lim_{N\to+\infty}\frac{1}{N}\sum_{n<N} f(T^{n+k}x)\cdot\overline{f(T^n x)}\cdot f'(S^{n+k}y)\cdot\overline{f'(S^n y)}$$

exists;

(b) *for ν-a.a. y, there exists a positive measure σ_{xy} on $[0,2\pi]$ such that*

$$L_k(x,y) = \int_0^{2\pi} e^{-itk}\sigma_{xy}(dt) \qquad (k\in\mathbb{Z});$$

(c) *if $f\in C^\perp$, then, for ν-a.a. y, the measure σ_{xy} is continuous.*

Proof:

(a) Just apply Bourgain's return-times theorem to the system $(X,\dots,T)$ with the function $f(T^kx)\cdot\overline{f(x)}$ and to the system $(Y,\dots,S)$ with the function $f'(S^ky)\cdot\overline{f'(y)}$.

(b) It is a direction application of Bochner's theorem. Note that

$$L_{k_1-k_2}(x,y)=\lim_{N\to+\infty}\frac{1}{N}\sum_{n<N} f(T^{n+k_1}x)\cdot\overline{f(T^{n+k_2}x)}\cdot f'(S^{n+k_1}y)\cdot\overline{f'(S^{n+k_2}y)}.$$

(c) We now suppose that $f\in\mathcal{C}^\perp$. By Theorem 3′, this implies that the function $f\otimes f'$ is orthogonal to the Kronecker factor for almost all ergodic components of the product measure $\mu\times\nu$. By the Wiener–Wintner theorem, this implies that, for $\mu\times\nu$-a.a. (x,y), for all $\alpha\in\mathbb{R}$,

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n<N} e^{in\alpha}\cdot f(T^nx)\cdot f'(S^ny)=0\,.$$

Using the notation and the first part of Theorem 4, we have:

$$(7)\quad\begin{cases}\text{for all } x\in\widetilde{X},\\ \text{for } \nu\text{-a.a. } y \text{ for } m_{xy}\text{-a.a. } (\tilde{x},\tilde{y}),\\ \text{for all } \alpha\in\mathbb{R},\ \lim\frac{1}{N}\sum_{n<N}e^{in\alpha}\cdot f(T^n\tilde{x})f'(S^n\tilde{y})=0.\end{cases}$$

By the second part of Theorem 4, there exists a set X'_f of full measure in X such that:

$$(8)\quad\begin{cases}\text{for all } x\in X'_f,\text{ for any ergodic m.p.s. } (Y,\varphi,\nu,S),\text{ for all } F\in C(Y),\\ \text{for all } k\in\mathbb{Z},\text{ and for } \nu\text{-a.a. } y,\\ \lim\frac{1}{N}\sum\limits_{n<N} f(T^{n+k}x)\overline{f(T^nx)}F(S^ny)=\int f(T^k\tilde{x})\cdot\overline{f(\tilde{x})}\cdot F(\tilde{y})dm_{xy}(\tilde{x},\tilde{y})\,.\end{cases}$$

We now consider an element x of $X'_f\bigcap\widetilde{X}$ which satisfies properties (a) and (b) of our proposition.

By (8), we have, for ν-a.a. y,

$$L_k(x,y)=\int f(T^k\tilde{x})\cdot\overline{f(\tilde{x})}\cdot f'(S^k\tilde{y})\cdot\overline{f'(\tilde{y})}dm_{xy}(\tilde{x},\tilde{y})\,.$$

If $\epsilon\in[0,2\pi]$, we have

$$\begin{aligned}\sigma_{xy}(\{\epsilon\})&=\lim_{N\to+\infty}\frac{1}{N}\sum_{k<N}e^{i\epsilon k}L_k(x,y)\\&=\lim_{N\to+\infty}\int\overline{f(\tilde{x})}\ \overline{f'(\tilde{y})}\Big(\frac{1}{N}\sum_{k<N}e^{i\epsilon k}\cdot f(T^k\tilde{x})\cdot f'(S^k\tilde{y})\Big)dm_{xy}(\tilde{x},\tilde{y})\\&=0\qquad\text{by (7).}\end{aligned}$$

This concludes the proof of Proposition 2. ∎

END OF THE PROOF OF THEOREM 1. We suppose that $f \in L^\infty(\mu) \cap \mathcal{C}^\perp$ and $f' \in C(Y)$. We want to prove that, for any x in the set of full measure given by Proposition 2, for ν-a.a. y,

$$\lim_{N\to+\infty} \sup_{\theta\in\mathbb{R}} \left| \frac{1}{N} \sum_{n<N} e^{in\theta} \cdot f'(S^n y) \cdot f(T^n x) \right| = 0 .$$

This can be achieved using Van der Corput's classical inequality. Let us recall it: if $(u_n)_{0\le n\le N-1}$ is a family of complex numbers and if K is an integer between 0 and $N-1$, then

$$\left| \frac{1}{N} \sum_{n<N} u_n \right|^2 \le \frac{N+K}{N^2(K+1)} \sum_{n<N} |u_n|^2 + \frac{2(N+K)}{N^2(K+1)^2} \sum_{k=1}^{K} (K+1-k) \operatorname{Re}\left(\sum_{n=0}^{N-k-1} u_{n+k} \cdot \bar{u}_n \right).$$

The proof of this inequality can be found, for example, in [KN]. Using this inequality, we have:

$$\begin{aligned} &\left| \frac{1}{N} \sum_{n<N} e^{in\theta} \cdot f'(S^n y) \cdot f(T^n x) \right|^2 \\ &\le \frac{N+K}{N^2(K+1)} \sum_{n<N} |f'(S^n y) f(T^n x)|^2 + \frac{2(N+K)}{N^2(K+1)^2} \\ &\cdot \sum_{k=1}^{K} (K+1-k) \operatorname{Re}\left(e^{ik\theta} \sum_{n=0}^{N-k-1} f(T^{n+k}x) \cdot \overline{f(T^n x)} \cdot f'(S^{n+k} y) \cdot \overline{f'(S^n y)} \right). \end{aligned}$$

We fix K. The first term of the right hand side of the last inequality satisfies

$$\limsup_{N\to+\infty} \frac{N+K}{N^2(K+1)} \sum_{n<N} |f'(S^n y) f(T^n x)|^2 \le \frac{\|f'\|_\infty^2 \, \|f\|_\infty^2}{K+1} .$$

In order to study the second term, we take x in the set of full μ-measure given by Proposition 2. We then have, for ν-a.a. y,

$$\limsup_{N\to+\infty} \sup_{\theta\in\mathbb{R}} \frac{2(N+K)}{N^2(K+1)^2}\sum_{k=1}^{K}(K+1-k)$$

$$\cdot \operatorname{Re}\left(e^{ik\theta}\sum_{n=0}^{N-k-1} f(T^{n+k}x)\cdot\overline{f(T^nx)}\cdot f'(S^{n+k}y)\cdot\overline{f'(S^ny)}\right)$$

$$= \sup_{\theta\in\mathbb{R}}\operatorname{Re}\left(\frac{2}{(K+1)^2}\sum_{k=1}^{K}(K+1-k)e^{ik\theta}\int_0^{2\pi} e^{-ikt}\sigma_{xy}(dt)\right).$$

We can conclude with the next lemma.

LEMMA: *If σ is a continuous measure on the interval, then*

$$\lim_{K\to+\infty}\int_0^{2\pi}\frac{1}{K^2}\sum_{k=1}^{K}(K+1-k)e^{ik(\theta-t)}\sigma(dt)=0 \qquad \textit{uniformly in } \theta.$$

And this lemma is a simple consequence of the following remark:

if σ is a continuous measure on the interval $[0,2\pi]$, then, for all $\delta>0$, there exist $\eta>0$ such that, for all intervals I of length less than η, we have $\sigma(I)<\delta$. ∎

5. Characterization of the algebra $\mathcal{C}$: proof of Theorem 2

The uniform version of the Wiener–Wintner return-times theorem for functions f in the orthogonal of $\mathcal{C}$ has been proved in the preceding section. This is the implication (i) $\Rightarrow$ (ii) of Theorem 2.

The fact that conditions (iii) and (iv) are necessary for this uniform property is, as we shall see, easy to prove. The interesting point is that each of these two conditions is sufficient.

Let $(X,\mathcal{F},\mu,T)$ be an ergodic m.p.s. and $f\in L^2(\mu)$. We begin by a remark showing that it is not possible to weaken the conditions (iii) and (iv) of Theorem 2.

Remark: We define two new conditions (iii)$'$ and (iv)$'$ by

(iii)$'$ for μ-almost all x, for all $\theta\in\mathbb{R}$, for μ-almost all x',

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{in\theta}\bar{f}(T^nx')f(T^nx)=0;$$

(iv)′

$$\lim_{N\to+\infty}\left\|\sup_{\theta\in\mathbb{R}}\left\|\frac{1}{N}\sum_{n=0}^{N-1}e^{in\theta}\bar{f}(T^nx')f(T^nx)\right\|_{L^2(d\mu(x'))}\right\|_{L^2(d\mu(x))}=0\,.$$

These two conditions are not sufficient to ensure that $f\in\mathcal{C}^\perp$. This can easily be justified looking at the dynamical system $(\mathbb{T}^2,(x,y)\mapsto(x+\alpha,y+x))$ where α is an irrational real number, and at the function $f(x,y)=\exp(2i\pi y)$. For this system, we have $\mathcal{C}^\perp=\{0\}$.

We are now going into some detail in the proof of Theorem 2.

In order to prove the implication (ii) $\Rightarrow$ (iii) and (iv), we are going to apply condition (ii) with $(Y,\varphi,\nu,S)=(X,\mathcal{F},\mu,T)$ and $f'=\bar{f}$. The implication (ii) $\Rightarrow$ (iii) is clear. For the other one, we remark that

$$\sup_\theta\left|\frac{1}{N}\sum_{n<N}e^{in\theta}\bar{f}(T^nx')f(T^nx)\right|^2\le\left(\frac{1}{N}\sum_{n<N}\left|\bar{f}(T^nx')\right|^2\right)\left(\frac{1}{N}\sum_{n<N}\left|f(T^nx)\right|^2\right),$$

and, by the mean ergodic theorem, the sequence on the right side of this inequality is convergent in $L^1(\mu\otimes\mu)$. This implies that the sequence on the left is uniformly integrable. So, if we know that it tends to zero almost everywhere, then it tends to zero in the mean. We have shown that (ii) $\Rightarrow$ (iv).

It remains to prove that (iii) or (iv) $\Rightarrow f\in\mathcal{C}^\perp$. Under the condition (iii) we have:

for μ-almost all ergodic components $(\mu\otimes\mu)_c$ of the product measure $\mu\otimes\mu$, for $(\mu\otimes\mu)_c$-almost all (x,x'), for all $\theta\in\mathbb{R}$,

$$\lim\frac{1}{N}\sum_{n<N}e^{in\theta}\bar{f}(T^nx')f(T^nx)=0\,.$$

By the Wiener–Wintner theorem, this implies that, for almost all ergodic components $(\mu\otimes\mu)_c$, the function $f\otimes\bar{f}$ is orthogonal to the Kronecker factor of the system $(X\times X,\ \mathcal{F}\times\mathcal{F},\ (\mu\otimes\mu)_c,\ T\times T)$.

Under condition (iv) we have: for almost all ergodic components $(\mu\otimes\mu)_c$, the sequence

$$\sup_\theta\left|\frac{1}{N}\sum_{n<N}e^{in\theta}\bar{f}(T^nx')f(T^nx)\right|$$

tends to zero in $L^2((\mu\otimes\mu)_c)$; and we know that if, for all θ, the sequence

$$\frac{1}{N}\sum_{n<N} e^{in\theta}\bar{f}(T^n x')f(T^n x)$$

tends to zero in $L^2((\mu\otimes\mu)_c)$, then the function $f\otimes\bar{f}$ is orthogonal to the Kronecker factor of the system $(X\times X,\ \mathcal{F}\times\mathcal{F},\ (\mu\otimes\mu)_c,\ T\times T)$.

Thus conditions (iii) and (iv) have the same consequence, and the next proposition will conclude the proof of Theorem 2. This proposition can be viewed as a converse to Theorem 3′.

PROPOSITION 3: *If, for almost all ergodic components $(\mu\otimes\mu)_c$ of the product measure $\mu\otimes\mu$ under the transformation $T\times T$, the function $f\otimes\bar{f}$ is orthogonal to the Kronecker factor of the system $(X\times X,\ \mathcal{F}\times\mathcal{F},\ (\mu\otimes\mu)_c),\ T\times T)$, then the function f is orthogonal to the $\mathcal{C}$-factor of the system $(X,\mathcal{F},\mu,T)$.*

This result appears in [R2]. We now give a short proof of it, using the Wiener–Wintner point of view. We use notation introduced in the first section. The $\mathcal{C}$-factor is $(G\times G_2,\ m:=m_G\otimes m_{G_2},\ R_{g_0,\varphi})$. We can write $f=f_1+f_2$ with $f_1\in\mathcal{C}$ and $f_2\in\mathcal{C}^{\perp}$. We know, by Theorem 3′, that $f_1\otimes\bar{f}_2,\ f_2\otimes\bar{f}_1$ and $f_2\otimes\bar{f}_2$ are orthogonal to the Kronecker factor in almost all m.p.s. $(X\times X,\ (\mu\otimes\mu)_c,\ T\times T)$.

We want to prove that if $f_1\neq 0$, then the function $f_1\otimes\bar{f}_1$ does not have this property. We are looking at the functionf_1 as defined on $G\times G_2$, and we suppose $f_1\neq 0$. We can write $f_1=\sum_{\mathcal{X}\in\hat{G}_2} a(\mathcal{X})\otimes\mathcal{X}$ in L^2 where, for each character $\mathcal{X}$ of G_2, the coefficient $a(\mathcal{X})$ is a square integrable function on G. The ergodic components of the product measure on $(G\times G_2)\times(G\times G_2)$ are indexed by $c\in G$ and are given by an isomorphism I_c of measure preserving systems:

$$((G\times G_2)\times(G\times G_2),(m\otimes m)_c,R_{g_0,\varphi}\times R_{g_0,\varphi})$$
$$\xleftarrow{I_c}(G\times G_2\times G_2,m_G\otimes m_{G_2}\otimes m_{G_2},U_c)$$

where $I_c(g,g_2,g_2')=(g,g_2,cg,g_2')$ and

$$U_c(g,g_2,g_2')=(g_0g,\varphi(g)\cdot g_2,\varphi(cg)\cdot g_2')\,.$$

We have

$$f_1\otimes\bar{f}_1=\sum_{\mathcal{X},\mathcal{X}'\in\hat{G}_2}(a(\mathcal{X})\otimes\mathcal{X})\otimes\left(\overline{a(\mathcal{X}')\otimes\mathcal{X}'}\right) \tag{9}$$

and this is an orthogonal decomposition in $L^2((m\otimes m)_c)$.

Choose $a(\mathcal{X})\neq 0$ and define $F:=(a(\mathcal{X})\otimes\mathcal{X})\otimes\overline{(a(\mathcal{X})\otimes\mathcal{X})}$ and $F_c:=F\circ I_c$. We have

$$F_c(g,g_2,g_2')=[a(\mathcal{X})(g)][(\overline{a(\mathcal{X})}(cg)]\mathcal{X}(g_2)\overline{\mathcal{X}(g_2')}$$

and

$$\begin{aligned}&\left|\frac{1}{N}\sum_{n<N}e^{in\theta}F_c\left(U_c^n(g,g_2,g_2')\right)\right|\\&\qquad=\left|\frac{1}{N}\sum_{n<N}e^{in\theta}[a(\mathcal{X})](g_0^ng)\cdot[a(\overline{\mathcal{X}})](cg_0^ng)\cdot\mathcal{X}(\varphi^{(n)}(g))\cdot\overline{\mathcal{X}(\varphi^{(n)}(cg))}\right|\\&\qquad=\left|\frac{1}{N}\sum_{n<N}e^{in\theta}[a(\mathcal{X})](g_0^ng)[a(\mathcal{X})](cg_0^ng)\cdot\overline{\lambda^n(c)}\;\overline{u(c,g_0^ng)}\right|\end{aligned}$$

(by the functional equation (E)). This expression has the form

$$\left|\frac{1}{N}\sum_{n<N}e^{in\theta(c)}A(c,g_0^ng)\right|$$

where, for a non-negligible set of c in G, the function $A(c,\cdot)$ is different from zero. For each such c, there exists $\theta\in\mathbb{R}$ with

$$\limsup_N\left\|\frac{1}{N}\sum_{n<N}e^{in\theta}A(c,g_0^ng)\right\|_{L^2(dg)}>0\,.$$

For a non-negligible set of c in G, there exists $\theta\in\mathbb{R}$ such that the sequence

$$\left\|\frac{1}{N}\sum_{n<N}e^{in\theta}F\circ\left(R_{g_0,\varphi}\times R_{g_0,\varphi}\right)^n\right\|_{L^2((m\otimes m)_c)}=\left\|\frac{1}{N}\sum_{n<N}e^{in\theta}F_c\circ U_c^n\right\|_{L^2}$$

does not tend to zero.

Using the orthogonal decomposition (9), we have: for a non-negligible set of c in G, there exist $\theta\in\mathbb{R}$ such that the sequence

$$\left\|\frac{1}{N}\sum_{n<N}e^{in\theta}\left(f_1\otimes\bar f_1\right)\circ\left(R_{g_0,\varphi}\times R_{g_0,\varphi}\right)^n\right\|_{L^2((m\otimes m)_c)}\quad\text{does not tend to zero.}$$

This means that, for a non-negligible set of c in G, the function $f_1\otimes\bar f_1$ is not orthogonal to the Kronecker factor of the m.p.s. $(X\times X,(m\otimes m)_c,T\times T)$. ∎

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